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Global Stability Analysis in Hopfield Neural Networks

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Abstract—In this paper, one approach is employed to investigate the existence and uniqueness of the equilibrium and the global attractivity of Hopfield neural network models. Without assuming the boundedness, monotonicity, and differentiability of the activation functions, by using M-matrix theory, Liapunov functions are constructed and employed to establish sufficient conditions for global asymptotic stability.

Keywords—Neural network, Global asymptotic stability, M-matrix.

1. INTRODUCTION

Hopfield [1] proposed the Hopfield-type neural networks as follows:

$$\frac{du(t)}{dt} = -Bu(t) + Ag(u(t)) + J, \tag{1}$$

where $u = (u_1, u_2, \dots, u_n)^\top$, $B = \operatorname{diag}(b_1, b_2, \dots, b_n)$, the $n \times n$ irreducible connection matrix $A = (a_{ij})$, $g(u) = (g_1(u_1), \dots, g_n(u_n))^\top$, and $J = (J_1, \dots, J_n)^\top$.

In applications to parallel computation and signal processing involving the solution optimization problems, it is required that system (1) has a unique equilibrium point that is globally attractive. Thus, the global attractivity of systems is of great importance for both practical and theoretical purposes and has been the major concern of most authors dealing with (1) and their generalizations [2–6].

Some results on global asymptotical stability previously quoted concern the case in which the neuron activations $g(u)$ are assumed bounded and strictly increasing (sigmoidal activations). Unfortunately, these assumptions make the results inapplicable to some important engineering problems [2,3,6]. In this paper, the conditions are relaxed to below.

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ASSUMPTION (A). *For each $j \in \{1, 2, \ldots, n\}$, $g_j : R \to R$ is globally Lipschitz with Lipschitz constant L_j; i.e., $|g_j(u_j) - g_j(v_j)| \le L_j|u_j - v_j|$ for all u_j, v_j.*

In this paper, without assuming the boundedness, monotonicity, and differentiability of the activation functions, by using M-matrix theory, Liapunov functions are constructed and employed to establish sufficient conditions for global asymptotic stability. We present new conditions ensuring existence, uniqueness, and globally asymptotical stability of the equilibrium point for a large class of neural networks.

For convenience, we introduce some notations. $x = (x_1, \ldots, x_n)^\top \in R^n$ denotes a column vector (the symbol $(\top)$ denotes transpose). $|x|$ denotes the absolute-value vector given by $|x| = (|x_1|, \ldots, |x_n|)^\top$. For matrix $A = (a_{ij})_{n\times n}$, $A^\top$ denotes the transpose of A, A^{-1} denotes the inverse of A, $[A]^s$ is defined as $[A]^s = (A^\top + A)/2$, and $|A|$ denotes absolute-value matrix given by $|A| = (|a_{ij}|)_{n\times n}$. If A is a symmetric matrix, $A > 0$ $(A \ge 0)$ means that A is positive definite (positive semidefinite). If A, B are symmetric matrices, $A > B$ $(A \ge B)$ means that $A - B$ is positive definite (positive semidefinite). By $\|x\|$, we denote a vector norm defined by $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$, while $\|A\|$ denotes a matrix norm defined by $\|A\| = (\max\{\lambda : \lambda \text{ is an eigenvalue of } A^\top A\})^{1/2}$. $\dot{V}$ denotes time derivative of V.

2. EXISTENCE AND UNIQUENESS OF THE EQUILIBRIUM

It is known that the equilibria of (1) are the solutions of the nonlinear equations associated with (1) as follows:

$$-Bu + Ag(u) + J = 0. \tag{2}$$

DEFINITION 1. *A real $n\times n$ matrix $A = (a_{ij})$ is said to be an M-matrix if $a_{ij} \le 0$, $i, j = 1, 2, \ldots, n$, $i \ne j$, and all successive principal minors of A are positive.*

LEMMA 1. *(See [5].) Let $A = (a_{ij})$ be an $n \times n$ matrix with nonpositive off-diagonal elements. Then the following statements are true.*

(i) *A is an M-matrix if and only if the real parts of all eigenvalues of A are positive.*
(ii) *A is an M-matrix if and only if there exists a vector $\xi > 0$ such that $\xi^\top A > 0$.*
(iii) *A is an M-matrix if and only if there exists a positive $n \times n$ diagonal matrix D such that matrix $AD + DA^\top$ is positive definite.*
(iv) *A is an M-matrix if and only if A is nonsingular and all elements of A^{-1} are nonnegative.*

To prove existence and uniqueness of the equilibrium point, we make use of these concepts from topology.

DEFINITION 2. *A map $H : R^n \to R^n$ is a homeomorphism of R^n onto itself if $H \in C^0$, H is one-to-one, H is onto and the inverse map $H^{-1} \in C^0$.*

LEMMA 2. *(See [6].) If $H(x) \in C^0$ satisfies the following conditions:*

(1) *$H(x)$ is injective on R^n,*
(2) *$\lim_{\|x\|\to\infty} \|H(x)\| \to \infty$,*

then $H(x)$ is a homeomorphism of R^n.

From Lemma 1, it is easy to obtain the following result.

LEMMA 3. *Let $Q = (q_{ij})_{n\times n}$ be an $n \times n$ matrix and B, L be $n \times n$ positive definite diagonal matrices; i.e., $B = \operatorname{diag}(b_1, \ldots, b_n)$ $(b_i > 0,\ i = 1, \ldots, n)$, $L = \operatorname{diag}(L_1, \ldots, L_n)$ $(L_i > 0,\ i = 1, \ldots, n)$. If $Q + BL^{-1}$ is an M-matrix, then for each diagonal matrix $K = \operatorname{diag}(k_1, \ldots, k_n)$ such that $0 \le K \le L$, we have that $QK + B$ is an M-matrix, and*

$$\det(QK + B) \ne 0, \tag{3}$$

where $0 \le K \le L$ implies $0 \le k_i \le L_i$ $(i = 1, 2, \ldots, n)$.

Define the following map associated with (2):

$$H(u) = -Bu + Ag(u) + J. \tag{4}$$

LEMMA 4. *If g satisfies Condition (A), and $\alpha = BL^{-1} - |A|$ is an M-matrix, then for every input J, the map H defined by (4) is injective.*

PROOF. Suppose, for purposes of contradiction, that there exist $x, y \in R^n$ with $x \neq y$ such that $H(x) = H(y)$. From (4), we get

$$-B(x-y) + A(g(x) - g(y)) = 0 \tag{5}$$

or

$$b_i(x_i - y_i) = \sum_{j=1}^{n} a_{ij}[g_j(x_j) - g_j(y_j)] \qquad (i = 1, 2, \dots, n).$$

Taking absolute values for both sides of the above, we obtain

$$b_i|x_i - y_i| \le \sum_{j=1}^{n} |a_{ij}|\,|g_j(x_j) - g_j(y_j)| \qquad (i = 1, 2, \dots, n). \tag{6}$$

From Condition (A), there exist $0 \le k_j \le L_j$ such that $|g_j(x_j) - g_j(y_j)| = k_j|x_j - y_j|$ $(j = 1, 2, \dots, n)$. So (6) becomes

$$b_i|x_i - y_i| \le \sum_{j=1}^{n} |a_{ij}|\,k_j\,|x_j - y_j| \qquad (i = 1, 2, \dots, n). \tag{7}$$

Equation (7) can be written as matrix form

$$(B - |A|K)|x - y| \le 0, \tag{8}$$

where $K = \mathrm{diag}(k_1, \dots, k_n)$. If $(B - |A|K)|x - y| = 0$, let $Q = -|A|$, from Lemma 3, when $\alpha = BL^{-1} - |A|$ is an M-matrix, $\det(B - |A|K) \neq 0$, where K is a diagonal matrix satisfying $0 \le K \le L$. So $|x - y| = 0$, i.e., $x = y$, which is a contradiction. If $(B - |A|K)|x - y| \neq 0$, from (8), there exists vector c such that $(B - |A|K)|x - y| = c \neq 0$, where all elements of c are nonpositive. From Lemma 1, all elements of $(B - |A|K)^{-1}$ are nonnegative, so all elements of $|x - y| = (B - |A|K)^{-1}c$ are nonpositive and there exist some element of $|x - y|$ is negative. But it is impossible. So map H is injective. The proof is completed.

LEMMA 5. *Suppose g satisfies Condition (A), and $\alpha = BL^{-1} - |A|$ is an M-matrix. Then, for every input J, map H is homeomorphism on R^n.*

PROOF. Because of $\alpha = BL^{-1} - |A|$ being an M-matrix, from Lemma 4, $H(u)$ is an injective map on R^n. From Lemma 2, if when $\|u\| \to +\infty$, $\|H(u)\| \to +\infty$, then H is a homeomorphism on R^n. Let $\bar{H}(u) = Bu + A(g(u) - g(0))$. To show that H is a homeomorphism, it suffices to show that $\bar{H}(u)$ is a homeomorphism.

From Definition 1, it is easy to verify that $B - |A|L$ is an M-matrix. From Lemma 1, there exists a positive definite diagonal matrix $C = \mathrm{diag}(c_1, \dots, c_n)$ such that

$$[C(-B + |A|L)]^S \le -\varepsilon E_n < 0 \tag{9}$$

for sufficiently small $\varepsilon > 0$. E_n is the identity matrix. Calculating

$$\begin{aligned}
[Cu]^\top \bar{H}(u) &= [Cu]^\top[-Bu + A(g(u) - g(0))] \\
&= -u^\top CBu + u^\top CA(g(u) - g(0)) \\
&\le -|u|^\top CB|u| + |u^\top|\,C|A|\,|g(u) - g(0)| \\
&\le -|u|^\top CB|u| + |u^\top|\,C|A|\,L|u| \\
&= |u|^\top [C(-B + |A|\,L)]^s |u| \\
&\le -\varepsilon\|u\|^2.
\end{aligned} \tag{10}$$

From (10) and using Schwartz inequality, we get

$$\varepsilon\|u\|^2 \le \|C\|\,\|u\|\,\left\|\bar{H}(u)\right\|.$$

When $\|u\| \ne 0$, we have

$$\frac{\varepsilon\|u\|}{\|C\|} \le \left\|\bar{H}(u)\right\|. \tag{11}$$

So when $\|u\| \to +\infty$, $\|\bar{H}(u)\| \to +\infty$; i.e., $\|H(u)\| \to +\infty$. From Lemma 2, we know that for every input J, map H is a homeomorphism on R^n. The proof is completed.

From Lemma 5, we can obtain the sufficient condition for the existence and uniqueness of the equilibrium in Hopfield neural networks.

THEOREM 1. *Suppose g satisfies Condition (A), and $\alpha = BL^{-1} - |A|$ is an M-matrix. Then, for every input J, system (1) has a unique equilibrium u^*.*

PROOF. Lemma 5 ensures that H is a homeomorphism on R^n. Hence, there exists a unique point u^* such that $H(u^*) = 0$. The proof is completed.

THEOREM 2. *Suppose g satisfies Condition (A). If there exists a positive symmetric matrix P such that $Q = -(B^\top P + PB) \le -\mu E_n$ $(\mu > 0)$ is negative definite, and*

$$\delta = \mu - 2L_{\max}\|PA\| > 0, \qquad \left(L_{\max} = \max_{1\le j\le n}\{L_j\}\right), \tag{12}$$

then system (1) has a unique equilibrium u^*.

PROOF. It is only needed to prove that H is a homeomorphism on R^n. First, we will prove that H is injective on R^n. Suppose, for purposes of contradiction, that there exist $x, y \in R^n$ with $x \ne y$ such that $H(x) = H(y)$. From (4), we get

$$-B(x-y) + A(g(x) - g(y)) = 0, \tag{13}$$

or

$$b_i(x_i - y_i) = \sum_{j=1}^{n} a_{ij}[g_j(x_j) - g_j(y_j)] \qquad (i = 1, 2, \dots, n). \tag{14}$$

From Condition (A), there exist $-L_j \le k_j \le L_j$ such that $g_j(x_j) - g_j(y_j) = k_j(x_j - y_j)$ $(j = 1, 2, \dots, n)$. So (14) becomes

$$b_i(x_i - y_i) = \sum_{j=1}^{n} a_{ij}\, k_j(x_j - y_j) \qquad (i = 1, 2, \dots, n). \tag{15}$$

Equation (15) can be written as matrix form

$$(-B + AK)(x - y) = 0, \tag{16}$$

where $K = \operatorname{diag}(k_1, \dots, k_n)$. Now, we prove indirectly $\det(-B+AK) \ne 0$. Consider the following systems:

$$\dot{z} = (-B + AK)z. \tag{17}$$

Let $V(z) = z^\top P z$, calculating and estimating the derivative of $V(x)$ along (17) as follows:

$$\begin{aligned}\dot{V}(z) &= z^\top(-BP - PB + 2PAK)z \\ &\le (-\mu + 2k_{\max}\|PA\|)\,\|z\|^2 \\ &\le (-\mu + 2L_{\max}\|PA\|)\,\|z\|^2 < 0,\end{aligned}$$

where $k_{\max} = \max_{1 \le j \le n}\{k_j\}$. By the Liapunov stability theorem [7], the trivial solution of (17) is asymptotically stable, so $\det(-B + AK) \neq 0$ for $-L \le K \le L$. Therefore, $x = y$, which is a contradiction. So map H is injective.

In the following, we prove that when $\|u\| \to +\infty$, $\|H(u)\| \to +\infty$. Let $\bar{H}(x) = -Bu + A(g(u) - g(0))$. Calculate

$$\begin{aligned} 2u^{\top} P\bar{H} &= u^{\top} P\bar{H} + \bar{H}^{\top} Pu \\ &= -u^{\top}\left(B^{\top} P + PB\right) u + 2uPA(g(u) - g(0)) \\ &\le -\mu\|u\|^2 + 2\,\|PA\|\, L_{\max}\|u\|^2 \\ &= -\delta\|u\|^2. \end{aligned}$$

Therefore,

$$\delta\|u\|^2 \le 2\|u\|\ \|P\|\ \left\|\bar{H}(u)\right\|.$$

Simplifying the above inequality, we get

$$\frac{\delta\|u\|}{(2\|P\|)} \le \left\|\bar{H}(u)\right\|.$$

So when $\|u\| \to +\infty$, $\|\bar{H}(u)\| \to +\infty$; i.e., $\|H(u)\| \to +\infty$. From Lemma 2, we know that for every input J, map H is a homeomorphism on R^n. The proof is completed.

3. GLOBAL ASYMPTOTIC STABILITY OF THE EQUILIBRIUM POINT

THEOREM 3. *Suppose (A) holds. If $\alpha = BL^{-1} - |A|$ is an M-matrix, then for each $J \in R^n$, (1) has a unique equilibrium point, which is globally asymptotically stable.*

PROOF. Since $\alpha = BL^{-1} - |A|$ is an M-matrix, from Theorem 1, system (1) has a unique equilibrium point u^*. By means of coordinate translation $x(t) = u(t) - u^*$, (1) can be written as

$$\frac{dx_i(t)}{dt} = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \qquad (i = 1, \dots, n), \tag{18}$$

where $f_j(x_j) = g_j(x_j + u_j^*) - g_j(u_j^*)$ $(j = 1, \dots, n)$. System (18) has a unique equilibrium at $x = 0$.

Clearly, u^* is globally asymptotically stable for (1) if and only if the trivial solution of (18) is globally asymptotically stable.

Due to $\alpha = BL^{-1} - |A|$ being an M-matrix, from Lemma 1, there exist $\xi_i > 0$ $(i = 1, 2, \dots, n)$ such that

$$-\xi_i b_i + \sum_{j=1}^{n} \xi_j |a_{ji}| L_i < 0 \qquad (i = 1, 2, \dots, n). \tag{19}$$

Consider a Liapunov functional $V(x)$ defined by

$$V(x) = \sum_{i=1}^{n} \xi_i |x_i|. \tag{20}$$

Calculating the upper right derivative D^+V of V along the solutions of (18),

$$\begin{aligned} D^+V(x) &= \sum_{i=1}^{n} \xi_i \left\{ \operatorname{sgn} x_i \left[-b_i x_i + \sum_{j=1}^{n} a_{ij} f_j(x_j) \right] \right\} \\ &\leq \sum_{i=1}^{n} \xi_i \left\{ -b_i |x_i| + \sum_{j=1}^{n} |a_{ij}| \, |f_j(x_j)| \right\} \\ &\leq \sum_{i=1}^{n} \xi_i \left\{ -b_i |x_i| + \sum_{j=1}^{n} |a_{ij}| \, L_j |x_j| \right\} \\ &= \sum_{i=1}^{n} \left\{ -b_i \xi_i + \sum_{j=1}^{n} |a_{ji}| \, L_i \xi_j \right\} |x_i| < 0 \qquad (\|x\| \neq 0). \end{aligned} \tag{21}$$

By the Liapunov stability theorem [7], the trivial solution of (18) is globally asymptotically stable, and therefore, u^* is global asymptotically stable for (1). The proof is completed.

Conditions in Theorem 3 are explicit form, and hence, are convenient to verify in practice. But they have the disadvantage of neglecting the signs of entries in the connection matrix A, and thus, differences between excitatory and inhibitory effects might be ignored. In general, this is overly restrictive.

Now we use the above to establish the following result in terms of the spectral norm.

THEOREM 4. *Suppose g satisfies Condition (A). If there exists a positive symmetric matrix P such that $Q = -(BP + PB) \leq -\mu E_n$ ($\mu > 0$) is negative definite, and*

$$\delta = \mu - 2L_{\max}\|PA\| > 0, \tag{22}$$

then for every input J, system (1) has a unique equilibrium u^* that is globally asymptotically stable.

PROOF. From Theorem 2, for every input J, system (1) has a unique equilibrium u^*. By means of coordinate translation $x(t) = u(t) - u^*$, (1) can be written as

$$\frac{dx(t)}{dt} = -Bx(t) + AF(x), \tag{23}$$

where $F(x) = (f_1(x), \dots, f_n(x))^\top$ is defined by

$$F_j(x) = g_j\left(x_j(t) + u_j^*\right) - g_j(u_j^*). \tag{24}$$

Now, consider the Liapunov function $V(x) = x^\top Px$. Then along (23)

$$\begin{aligned} \dot{V}(x) &= -x^\top \left(B^\top P + PB\right) x + 2xPAF(x) \\ &\leq -\mu\|x\|^2 + 2\|x\| \, \|PA\| \, \|F(x)\|. \end{aligned} \tag{25}$$

From (A), we obtain

$$\|F(x)\|^2 = \sum_{j=1}^{n} f_j^2(x) \leq \sum_{j=1}^{n} L_j^2 x_j^2 \leq L_{\max}^2 \, \|x\|^2.$$

Thus,

$$\dot{V}(x) \leq (-\mu + 2\|PA\| \, L_{\max}) \, \|x\|^2 = -\varepsilon\|x\|^2.$$

By the Liapunov stability theorem [7], the trivial solution of (23) is globally asymptotically stable, and therefore, u^* is globally asymptotically stable for (1). The proof is completed.

REFERENCES

1. J.J. Hopfield, Neurons with graded response have collective computational properties like those of two sate neurons, *Proc. Natl. Acad. Sci. USA* **81**, 3088–3092, (1984).
2. M.P. Kennedy and L.O. Chua, Neural networks for nonlinear programming, *IEEE Trans. Circuits Syst.* **35**, 554–562, (1988).
3. D. Tank and J.J. Hopfield, Simple neural optimization networks: An A/D converter, signal decision circuit and a linear programming circuit, *IEEE Trans. Circuits and Systems* **33**, 533–541, (1986).
4. M. Forti, On global asymptotic stability of a class of nonlinear systems arising in neural network theory, *J. Differential Equations* **113**, 246–264, (1994).
5. A.N. Michel, J.A. Farrell and W. Porod, Qualitative analysis of neural networks, *IEEE Trans. on Circuits and Systems* **36**, 229–243, (1989).
6. M. Forti and A. Tesi, New conditions for global stability of neural networks with application to linear and quadratic programming problems, *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications* **42**, 354–366, (1995).
7. N. Rouche, P. Habats and M. Paloy, *Stability Theory by Liapunov's Direct Method*, Springer-Verlag, New York, (1977).